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The Use of Restricted Latent Class Models for Defining and Testing Nonparametric and Parametric Item Response Theory Models

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A general class of ordinal logit models is presented that specifies equality and inequality constraints on sums of conditional response probabilities. Using these constraints in latent class analysis, models are obtained that are similar to parametric and nonparametric item response models. Maximum likelihood is used to estimate these models, making their assumptions testable with likelihood-ratio statistics. Because of the intractability of the asymptotic distribution of the goodness-of-fit measure when imposing inequality constraints, parametric bootstrapping is used to obtain estimates of p values. The proposed restricted latent class models are illustrated by an example using reported adult crying behavior.
Index terms: inequality constraints, nonparametric item response theory, order-restricted inference, parametric bootstrapping, polytomous item response theory, restricted latent class analysis, stochastic ordering.

Two similarities exist between latent class models (LCMs) and item response theory (IRT) models. One is that order-restricted LCMs can be used to estimate nonparametric IRT models with maximum likelihood (ML; Croon, 1990, 1991) and Bayesian methods (Hoijtink & Molenaar, 1997). These methods specify simple inequality restrictions on the cumulative conditional response probabilities [item step response functions (ISRFs)]. There is also a similarity between loglinear LCMs and parametric IRT models (Heinen, 1996)—discretized variants of the most important parametric IRT models can be obtained by placing constraints on the loglinear parameters of LCMs.

Purpose

This paper integrates and extends these developments using a general class of loglinear equality and inequality constraints on ISRFs. The approach is based on work in generalized loglinear modeling (Bergsma, 1997; Lang & Agresti, 1994) and order-restricted inference with categorical variables (e.g., Dardanoni & Forcina, 1998; Robertson, Wright, & Dykstra, 1988; Vermunt, 1999). Restrictions of these forms can be used not only to specify nonparametric IRT models (e.g., the monotone homogeneity model for polytomous items; Molenaar, 1997), but also to define nonparametric variants of most parametric IRT models for polytomous items. Hybrid IRT models combining parametric with nonparametric features also can be obtained.

Ordinal Dependent and Independent Variable Models

Four Types of Odds/ISRFs

An empirical example taken from the Adult Crying Inventory (Becht, Poortinga, & Vingerhoets, in press) illustrates the logit models for ordinal independent and dependent variables. Table 1 is

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a two-way cross-tabulation of two ordinal variables—one from the "Crying From Distress" scale (collapsed into five categories of relatively equal size) and the questionnaire item "Feeling Relieved After Crying" (from a different scale), denoted X and Y, respectively. X contains I levels, and Y contains J levels, where $I = 5$ and $J = 3$. Their respective category indices are denoted i and j.

Table 1
Observed Cross-Classification of "Crying From Distress" and "Feeling Relieved After Crying" (1 = Low, 5 = High)

Crying From	Feeling Relieved After Crying		
Distress	Less	Same	More
1	61	195	438
2	78	158	581
3	38	102	518
4	46	119	572
5	53	106	597

Assume that X is the independent variable and Y the dependent variable. Interest is then on the conditional distribution of Y, given X. The question is then whether persons who more frequently cry experience more benefits from crying—whether there is a positive relationship between X and Y.

The standard way to model this relationship of ordinal variables is through a logit model that imposes equality constraints on odds ratios. For this purpose, four types of odds can be used (Agresti, 1990, Section 9.3; Mellenbergh, 1995):
cumulative odds,

$$\Omega_{i,j}^{cum} = P(Y \leq j-1|X=i)/P(Y \geq j|X=i) \ , \tag{1}$$

adjacent category (local) odds,

$$\Omega_{i,j}^{adj} = P(Y = j-1|X=i)/P(Y = j|X=i) \ , \tag{2}$$

continuation odds Type 1,

$$\Omega_{i,j}^{conI} = P(Y = j-1|X=i)/P(Y \geq j|X=i) \ , \tag{3}$$

or continuation odds Type 2,

$$\Omega_{i,j}^{conII} = P(Y \leq j-1|X=i)/P(Y = j|X=i) \ , \tag{4}$$

where $2 \leq j \leq J$ and $1 \leq i \leq I$. $\Omega_{i,j}$ is used below as a generic symbol referring to any of these odds.

Modeling a certain type of odds corresponds to modeling a certain ISRF type. Initially, it is assumed here that there is no latent trait, but instead conditioning is based on a summated scale score, X. The ISRFs corresponding to the four types of odds (Mellenbergh, 1995; van der Ark, 2001) are

$$ISRF_{i,j}^{cum} = P(Y \geq j|X=i) \ , \tag{5}$$

$$ISRF^{adj}_{i,j} = \frac{P(Y = j|X = i)}{P(Y = j|X = i) + P(Y = j - 1|X = i)} , \qquad (6)$$

$$ISRF^{conI}_{i,j} = \frac{P(Y \geq j|X = i)}{P(Y \geq j - 1|X = i)} , \qquad (7)$$

and

$$ISRF^{conII}_{i,j} = \frac{P(Y = j|X = i)}{P(Y \leq j|X = i)} . \qquad (8)$$

ISRF$^{cum}_{i,j}$, the cumulative odds, is used in graded response models (GRMs), ISRF$^{adj}_{i,j}$ is used in partial credit models (PCMs), and ISRF$^{conI}_{i,j}$ is used in sequential models. (For a more detailed explanation of these ISRFs, see van der Ark, 2001.)

ISRF$^{conII}_{i,j}$ differs from ISRF$^{conI}_{i,j}$ with respect to the order in which persons are assumed to evaluate the response variable categories. If $\Omega^{conI}_{i,j}$ is used, it is assumed that persons evaluate the response alternatives from low to high. If $\Omega^{conII}_{i,j}$ is used, it is assumed they are evaluated from high to low.

In practical research situations, a choice must be made between these four types of odds/ISRFs. A model must be specified for the type of odds/ISRF that best fits the assumed process underlying the responses. Here, all four types are used in order to illustrate the generality of the approach.

Parametric Models

A logit model that takes into account that the dependent and independent variables are ordinal is (Agresti, 1990)

$$\log \Omega_{i,j} = \alpha_j - \beta x_i , \qquad (9)$$

where

x_i is the fixed score assigned to category i of X,
α_j is the intercept of the logit model, and
β is the slope of the logit model.

In most cases, x_i assumes equal intervals (e.g., 1, 2, 3, 4), but other scoring schemes also can be used for X. The ISRF model corresponding to Equation 9 is

$$ISRF_{i,j} = \frac{\exp(\alpha_j + \beta x_i)}{1 + \exp(\alpha_j + \beta x_i)} . \qquad (10)$$

Note that the ISRFs are assumed to have equal slopes. This implies that, with ISRF$^{adj}_{i,j}$ for instance, a model similar to a PCM is obtained. However, X is an observed variable, not a latent trait. When equal-interval x_i is used, the model described in Equation 9 implies that the log-odds ratios between adjacent levels of X are assumed to be constant,

$$\log \Omega_{i,j}/\Omega_{i+1,j} = \log \Omega_{i,j} - \log \Omega_{i+1,j} = \beta , \qquad (11)$$

for all i and j. These differences between log odds do not depend on the values of X and Y; this can be expressed by the equality constraints

$$(\log \Omega_{i,j} - \log \Omega_{i+1,j}) - (\log \Omega_{i,j+1} - \log \Omega_{i+1,j+1}) = 0 , \qquad (12)$$

and

$$(\log \Omega_{i,j} - \log \Omega_{i+1,j}) - (\log \Omega_{i+1,j} - \log \Omega_{i+2,j}) = 0 \; . \quad (13)$$

The restrictions implied by standard ordinal logit models also can be defined in terms of equalities on the log-odds ratios—the model parameters α_j and β are eliminated (this is used in the constrained-optimization procedure described below).

Table 2 reports the observed log-cumulative-odds ratios for the data from Table 1. Note that a log-odds ratio larger than 0.0 is in agreement with the postulated positive relationship between X and Y. The data in Table 2 contain four violations of ordinal relationships. It should be determined whether these were the result of sampling fluctuation.

Table 2
Log-Cumulative-Odds Ratios of "Feeling Relieved After Crying" for Adjacent Categories and "Crying From Distress" (Collapsed Into Five Levels) Observed and Estimated Under the Constraints of Equation 14

	Feeling Relieved After Crying	
Crying From Distress	Less vs. Same/More	Less/Same vs. More
1 vs. 2	−.09/.00	.36/.39
2 vs. 3	.54/.40	.41/.34
3 vs. 4	−.08/.00	−.07/.00
4 vs. 5	−.12/.00	.08/.10

Table 3 reports test results for the estimated models. The independence model test statistic shows a significant association between the two variables. None of the logit models using the constraints (Equations 12 and 13) fit the data. Thus, the parametric assumptions of the standard ordinal logit models are too restrictive for this dataset.

Table 3
G^2, df, and p for Ordinal Logit Models Estimated With Crying Data for Parametric Models (Restrictions From Equations 12 and 13) and Nonparametric Models (Restrictions From Equation 14)

Model	G^2	df	p
Independence	76.6	8	.00
Parametric			
Cumulative (GRM)	23.2	7	.00
Adjacent (PCM)	34.0	7	.00
Continuation I (SRM)	21.7	7	.00
Continuation II (SRM)	39.6	7	.00
Nonparametric			
Cumulative (GRM)	1.3	4	.56
Adjacent (PCM)	7.3	5	.14
Continuation I (SRM)	1.3	4	.58
Continuation II (SRM)	7.3	5	.11

Nonparametric Models

The equalities implied by the above logit model can be replaced by inequalities. The approach then becomes nonparametric, rather than parametric. This yields a less-restrictive definition of a positive log-odds ratio relationship,

$$\log \Omega_{i,j} - \log \Omega_{i+1,j} \geq 0 \ . \tag{14}$$

It has been assumed that all log-odds ratios are at least 0.0. Such a set of constraints is referred to as: simple stochastic ordering; likelihood ratio ordering; or uniform stochastic ordering for cumulative, adjacent category, and continuation odds (Dardanoni & Forcina, 1998). The nonnegativity constraint on the log-odds ratios can also be formulated in terms of constraints on the ISRFs (van der Ark 2001). That is,

$$ISRF_{i,j} \leq ISRF_{i+1,j} \ . \tag{15}$$

These constraints are special cases of a more general class of equality and inequality constraints (see below). Hybrid ordinal logit models can be obtained by combining inequality and equality restrictions. For example, combining the restrictions in Equation 14 with those in Equation 13 yields a model with monotonically decreasing odds; the decrease is constant between adjacent values of X.

Models using the inequality constraints defined in Equation 14 were estimated for the empirical dataset. Table 3 gives results for the four order-restricted models. As can be seen, the order-restricted nonparametric models fit much better than the parametric logit models. Thus, there was no evidence against a monotonic relationship between X and Y. The choice between the four nonparametric models should not depend only on the fit of the models, but also on the plausibility of the assumed process generating the responses.

As shown in Table 2, using inequality constraints leads to certain log-odds ratios of 0.0. Similar results for the other three types of log-odds ratios are not presented.

LCMs for Ordinal Items

Unrestricted LCMs

Assume an LCM with a single latent trait X and K items denoted Y_k, where $1 \leq k \leq K$. The number of latent classes [levels of the (discretized) latent trait] is I, the number of levels of item Y_k is J_k, and i and j_k are specific levels of X and Y_k, respectively.

In an LCM, it is assumed that items are independent of each other within latent classes (i.e, local independence; see, e.g., Bartholomew & Knott, 1999; Goodman, 1974). An LCM with a single latent variable is

$$P(Y_1 = j_1, Y_2 = j_2, \ldots, Y_K = j_K) = \sum_{i=1}^{I} P(X = i) \prod_{k=1}^{K} P(Y_k = j_k | X = i) \ , \tag{16}$$

where $P(X = i)$ is the unspecified distribution of the latent trait, and $P(Y_k = j_k | X = i)$ are the item response probabilities. In a standard LCM, no restrictions are imposed on these probabilities.

Parametric Models

To obtain a monotone relationship between X and Y_k, the logit constraints described above are used. For example, the equality restrictions in Equations 12 and 13 can be imposed on the ISRFs obtained with $P(Y_k = j_k | X = i)$. Denoting the odds for item k by $\Omega^k_{i,j}$, these equality constraints are

$$\left(\log \Omega^k_{i,j} - \log \Omega^k_{i+1,j}\right) - \left(\log \Omega^k_{i,j+1} - \log \Omega^k_{i+1,j+1}\right) = 0 \ , \tag{17}$$

and

$$\left(\log \Omega^k_{i,j} - \log \Omega^k_{i+1,j}\right) - \left(\log \Omega^k_{i+1,j} - \log \Omega^k_{i+2,j}\right) = 0 \ . \tag{18}$$

Equation 17 results in category-independent odds ratios for item k, and Equation 18 makes them class independent.

Heinen (1996, pp. 120–133) showed that an LCM for polytomous items with restrictions on the adjacent category log-odds ratios (such as those in Equation 18) yields a model similar to the nominal response model (Bock, 1972). Heinen also demonstrated that when the constraints in Equations 17 and 18 are used at the same time, a discretized variant of the PCM (Masters, 1982) with item-specific slopes is obtained. This is usually referred to as the generalized PCM (Muraki, 1992). These constraints imply that

$$ISRF^k_{i,j} = \frac{\exp(\alpha^k_j + \beta^k x_i)}{1 + \exp(\alpha^k_j + \beta^k x_i)} \ . \tag{19}$$

These types of constraints can be used on the cumulative and continuation odds as well as the adjacent category odds. This yields discretized variants of the GRM (Samejima, 1969) and the sequential response model (SRM; Mellenbergh, 1995; Tutz, 1990). The only difference between a parametric IRT model and the corresponding restricted LCM is that the distribution of the trait is described by a small number of points (classes) with unknown weights (sizes), rather than by a known distributional form.

Parametric IRT models are often estimated using marginal ML, in which normal quadratures are used to solve the integrals appearing in the likelihood function. This implicitly assumes a discretized latent variable. Rasch models are usually estimated by conditional ML, which involves conditioning on the (discrete) total score. Lindsay, Clogg, & Grego (1991) showed that equivalent estimates can be obtained for the Rasch model and a restricted LCM.

Nonparametric Models

Monotonicity can also be obtained using the inequality constraints in Equation 14 on the various log-odds ratios. When translated into the LCM context, these become

$$\log \Omega^k_{i,j} - \log \Omega^k_{i+1,j} \geq 0 \ , \tag{20}$$

which indicates that $\Omega^k_{i,j}$ decreases or remains equal as i increases. Croon (1990, 1991) and Hoijtink & Molenaar (1997) proposed using such constraints with cumulative odds, yielding a discretized variant of the nonparametric GRM (Hemker, Sijtsma, Molenaar, & Junker, 1997), also known as the polytomous monotone homogeneity model (Molenaar, 1997). The more general approach presented here allows the use of nonnegative log-odds ratio restrictions on the adjacent category and continuation odds, yielding nonparametric variants of the PCM and SRM.

Models With Between-Item Constraints

In addition to the restrictions within items on ISRFs, it might be relevant to use equality or inequality constraints between items. The most relevant inter-item equality restriction is

$$\left(\log \Omega^k_{i,j} - \log \Omega^k_{i+1,j}\right) - \left(\log \Omega^\ell_{i,j} - \log \Omega^\ell_{i+1,j}\right) = 0 \ , \tag{21}$$

which essentially equates the discrimination parameters of items k and ℓ. That is, $\beta^k = \beta^\ell$. The most relevant inequality constraint across items is

$$\log \Omega^k_{i,j} - \log \Omega^\ell_{i,j} \geq 0 \,, \tag{22}$$

which specifies that item k is easier than item ℓ for latent class i. Using these constraints on all item pairs (k, ℓ) in combination with the inequality constraints described in Equation 20 yields a more-restricted variant of the polytomous monotone homogeneity model (the model of strong double monotonicity; Sijtsma & Hemker, 1998).

Extensions

It is also possible to combine parametric and nonparametric features. For example, combining Equation 20 with Equation 17 yields a model with category-independent odds ratios combined with ordered latent classes. When combined with adjacent-category odds, such a hybrid model has the form of an ordered-restricted row-association structure (Agresti, Chuang, & Kezouh, 1986; Vermunt, 1999).

Several interesting extensions are straightforward within the latent class framework developed here. The most important are models with several latent traits, models with covariates, and models with local dependencies. Each of these extensions can be implemented using the framework introduced here.

Model Estimation and Testing

Equality Constraints

Parameter estimation that is less straightforward than ML estimation of standard ordinal logit models is useful when working with inequality constraints. The method used is based on generalized loglinear modeling (also known as marginal modeling; Bergsma, 1997; Lang & Agresti, 1994).

The estimation of the probabilities $P(Y = j|X = i)$ or $\pi_{j|i}$ is defined as a restricted optimization problem. The rth restriction on $\pi_{j|i}$ is

$$\sum_t c_{rt} \log \sum_{ij} a_{ijt}\pi_{j|i} = 0 \,, \tag{23}$$

where

- a_{ijt}, taking on the value of 1 or 0, defines the appropriate sums of probabilities on which the odds are based,
- t is the index used to denote the tth sum, and
- c_{rt} defines the relevant linear restrictions on the log-odds ratios (e.g., those in Equations 12 and 13).

Assuming a multinomial sampling scheme, ML estimation involves finding the saddle point of the following Lagrange equation (Gill & Murray, 1974)

$$\mathcal{L} = \sum_{ij} n_{ij} \log \pi_{j|i} + \sum_i \gamma_i \left(\sum_j \pi_{j|i} - 1 \right) + \sum_r \lambda_r \left(\sum_t c_{rt} \log \sum_{ij} a_{ijt}\pi_{j|i} \right) \,, \tag{24}$$

where γ_i and λ_r are Lagrange multipliers, and n_{ij} is an observed cell frequency.

Bergsma (1997) and Lang & Agresti (1994) provided two slightly different versions of the Fisher-scoring algorithm to solve this problem. Vermunt (1999) proposed a simple unidimensional Newton method that can be used for a more limited class of restrictions. For more general information on algorithms for constrained optimization, see Gill & Murray (1974).

Inequality Constraints

Estimation with inequality constraints is very similar to that with equality constraints. The sth inequality constraint is

$$\sum_t d_{st} \log \sum_{ij} a_{ijt} \pi_{j|i} \geq 0 \ . \tag{25}$$

ML estimation involves finding the saddle point of

$$\mathcal{L} = \sum_{ij} n_{ij} \log \pi_{j|i} + \sum_i \gamma_i \left(\sum_j \pi_{j|i} - 1 \right) + \sum_r \lambda_r \left(\sum_t c_{rt} \log \sum_{ij} a_{ijt} \pi_{j|i} \right) + \sum_s \delta_s \left(\sum_t d_{st} \log \sum_{ij} a_{ijt} \pi_{j|i} \right) \ , \tag{26}$$

where $\delta_s \geq 0$, and δ_i, λ_r, and γ_t are Lagrange multipliers.

The only difference between the situation with inequality constraints and that with equality constraints is that the Lagrange multipliers belonging to the inequality constraints must be at least 0.0. An inequality constraint is activated only if it is violated.

In practice, estimation can be accomplished by transforming the Fisher-scoring algorithm (Bergsma, 1997; Lang & Agresti, 1994) into an active-set method. Vermunt (1999) showed how to transform a simple unidimensional Newton algorithm for ML estimation with equality constraints into an active-set method. At each iteration cycle in active-set methods, the inequality restrictions that are no longer necessary (i.e., if $d_s < 0$) are deactivated. Those that are violated are activated. More general information on algorithms for optimization under equality and inequality constraints can be found in Gill & Murray (1974).

Restricted LCMs

Let $\pi^k_{j_k|i}$ be $P(Y_k = j_k | X = i)$. In latent class analysis, the equality and inequality restrictions used are special cases of the general form

$$\sum_k \sum_t c^k_{rt} \log \sum_{ij_k} a_{ij_kt} \pi^k_{j_k|i} = 0 \ , \tag{27}$$

and

$$\sum_k \sum_t d^k_{st} \log \sum_{ij_k} a_{ij_kt} \pi^k_{j_k|i} \geq 0 \ , \tag{28}$$

where

a_{ij_kt} specifies the relevant sums of probabilities, making it straightforward to switch from one type of log odds to the other;

c^k_{rt} is the linear equality constraint on logs of sums of probabilities; and

d^k_{st} is the inequality constraint on logs of sums of probabilities.

The first sum over items makes it possible specify between-item constraints.

Estimation can be performed by implementing the Fisher-scoring active-set method in the maximization (M) step of an EM algorithm (Dempster, Laird, & Rubin, 1977). This is similar to what Croon (1990, 1991) did with a pooling adjacent violators algorithm, which is a method for dealing with certain types of inequality restrictions. Vermunt (1999) proposed an EM algorithm which implements an active-set algorithm based on a unidimensional Newton algorithm in the M step.

An advantage of model estimation with the EM algorithm is that in the M step, the same types of estimation methods can be used as if the latent variable were observed. The expectation (E) step of the EM algorithm is very simple in LCMs.

Computer time for model estimation generally is not a problem for models with large numbers of items (e.g., 50 or 100)—estimation never takes more than a few minutes, as with parametric IRT models. It is well-known that the loglikelihood function of LCMs might be multimodal; this problem typically becomes worse when imposing inequality constraints. A practical solution is to run the same model with multiple (e.g., 10) sets of random starting values. Within a bootstrap (see below), the best procedure seems to be to start the estimation from the ML estimates.

Model Testing

Let H_1 be the hypothesized order-restricted model and H_0 the more restrictive model obtained by transforming all inequality restrictions into equality restrictions. These could, for instance, be nonnegative log-odds ratios (H_1) and independence (H_0). Whether H_1 fits the data can be tested using a standard likelihood-ratio statistic,

$$G^2 = 2 \sum_{ij} n_{ij} \ln \left(\frac{\hat{\pi}_{j|i}}{p_{j|i}} \right) , \tag{29}$$

where $\hat{\pi}_{j|i}$ and $p_{j|i}$ denote estimated and observed probabilities, respectively. A complication in using this test statistic is, however, that it is not asymptotically χ^2 distributed. It has been shown that it follows a $\bar{\chi}^2$ (rather than a χ^2) distribution, which are weighted sums of χ^2 distributions when H_0 holds (e.g., Robertson et al., 1988, p. 321).

Let S denote the number of inequality constraints, which is also the maximum number of activated constraints. The p value can be estimated as

$$P(G^2 \geq c) = \sum_{s=0}^{S} P(s) P \left[\chi^2_{(s)} \geq c \right] , \tag{30}$$

that is, as a weighted sum of asymptotic p values where the probability of having s activated constraints, $P(s)$, serves as a weight. This shows that it must be taken into account that the number of activated constraints is a random variable. A problem associated with Equation 30, however, is that the computation of the $P(s)$s is—except for some trivial cases—extremely complicated.

The p values for the test statistic can be estimated using parametric bootstrapping based on monte carlo studies. This relatively simple method, which involves empirically reconstructing the sampling distribution of the test statistic of interest, is followed here. Ritov & Gilula (1993) proposed such a procedure in ML correspondence analysis with ordered category scores. Their simulation study showed that parametric bootstrapping yields reliable results when applied in these models, which are special cases of the order-restricted LCMs presented here. Langeheine, Pannekoek, & van de Pol (1996) proposed using bootstrapping in categorical data analysis for dealing with sparse tables, which is another situation in which asymptotic theory cannot be relied upon for the test statistics. Agresti & Coull (1996) used monte carlo studies in combination with exact tests to determine the goodness of fit of order-restricted binary logit models estimated with small samples.

In the parametric bootstrap procedure, T frequency tables with the same number of observations as the original observed table are simulated from the estimated probabilities under H_1. For each of these tables, H_1 is estimated and the value of G^2 is computed. This yields an empirical approximation of the distribution of G^2. The estimated p value is the proportion of simulated tables with a G^2 that is at least as large as for the original table. The standard error of the estimated p value is $\sqrt{p(1-p)/T}$.

Example Application

Four items from "Crying From Distress" (Becht et al., in press) were used to illustrate the various types of LCMs for ordinal items: (1) "I cry when I feel frightened," (2) "I cry when I am in despair," (3) "I cry when I feel rejected by others," and (4) "I cry when I feel that I am in a blind-alley situation." The original seven-point scale (where 1 = never and 7 = always) was collapsed into three levels: 1–2, 3–5, and 6–7. Data were available for 3,821 respondents. p values were estimated using 1,000 bootstrap samples, with $T = 1{,}000$.

Table 4 reports the test results for the estimated unrestricted and restricted LCMs. As shown by the goodness-of-fit tests, the unrestricted four-class model best fit the data. The difference between G^2 values (likelihood-ratio statistics) for the three- and four-class models ($\Delta G^2 = 33.4$) was clearly significant with an estimated bootstrap p value of 0.0. The four-class model contained only a few order violations—an indication that it captured a single dimension in the items. The four-class model was retained as a basis for testing the validity of the restrictions corresponding to parametric and nonparametric IRT models.

Table 4 also shows test results for the four discretized parametric IRT models obtained by specifying four-class models with the equality constraints in Equations 17 and 18 on the four types of odds. This amounted to restricting the ISRFs to the same slope (see Equation 19). The reported p values show that none of these models fit the data well, indicating that the constraints implied by parametric IRT models were too restrictive for this dataset.

Table 4 reports the goodness-of-fit measures for four-class nonparametric LCMs with the inequality constraints in Equation 20. [For the order-restricted models, the reported degrees of freedom (df) were 45 plus the number of activated constraints.] Each fit the data at the .05 level. The GRM G^2 was lowest, the PCM G^2 value highest, and G^2 for the two SRMs were in the mid-range. This was expected because of the hierarchy among the inequality constraints: the inequality constraints on the adjacent category odds imply the inequality constraints on the two types of continuation odds and the cumulative odds, and the inequality constraints on one of the two types of continuation odds imply the inequality constraints on the cumulative odds (Hemker et al., 1997; van der Ark, 2001).

The double monotonicity models based on Equations 20 and 22 (nonparametric four-class models) fit the data well (all $p > .05$). In the GRM, no additional constraints are activated by the data compared to the model of monotone homogeneity. This is seen in the reported df: they were equal for the two nonparametric GRMs. When applying the strong double monotonicity restrictions on the adjacent category and continuation odds, additional constraints are activated by the data.

Table 4
G^2, df, and p for LCMs Estimated With "Crying From Distress" Data for Unrestricted Models, Parametric Four-Class Models (Restrictions of Equations 17 and 18), Nonparametric Four-Class Models (Restrictions of Equation 20), and Nonparametric Four-Class Models (Restrictions of Equations 20 and 22)

Model	G^2	df	p
Unrestricted			
One class	4597.5	72	.00
Two class	871.2	63	.00
Three class	86.8	54	.01
Four class	53.4	45	.28
Parametric four-class			
Cumulative (GRM)	187.2	65	.00
Adjacent (PCM)	223.9	65	.00
Continuation I (SRM)	202.3	65	.00
Continuation II (SRM)	212.5	65	.00
Nonparametric four-class			
Cumulative (GRM)	56.4	48	.28
Adjacent (PCM)	69.2	52	.07
Continuation I (SRM)	61.7	51	.23
Continuation II (SRM)	60.2	47	.17
Nonparametric four-class (double monotonicity)			
Cumulative (GRM)	56.4	48	.30
Adjacent (PCM)	69.8	53	.08
Continuation I (SRM)	62.0	53	.24
Continuation II (SRM)	60.6	50	.16

References

Agresti, A. (1990). *Categorical data analysis.* New York: Wiley.

Agresti, A., Chuang, C., & Kezouh, A. (1986). Order-restricted score parameters in association models for contingency tables. *Journal of the American Statistical Association, 82,* 619–623.

Agresti, A., & Coull, B. A. (1996). Order-restricted tests for stratified comparisons of binomial proportions. *Biometrics, 52,* 1103–1111.

Bartholomew, D. J., & Knott, M. (1999). *Latent variable models and factor analysis.* London: Arnold.

Becht, M. C., Poortinga, Y. H., & Vingerhoets, A. J. J. M. (in press). Crying across countries. In A. J. J. M. Vingerhoets & R. R. Cornelius (Eds.), *Adult crying: A biopsychosocial approach.* Reading, UK: Harwood.

Bergsma, W. P. (1997). *Marginal models for categorical data.* Tilburg, The Netherlands: Tilburg University Press.

Bock, R. D. (1972). Estimating item parameters and latent ability when responses are scored in two or more nominal categories. *Psychometrika, 37,* 29–51.

Croon, M. A. (1990). Latent class analysis with ordered latent classes. *British Journal of Mathematical and Statistical Psychology, 43,* 171–192.

Croon, M. A. (1991). Investigating Mokken scalability of dichotomous items by means of ordinal latent class analysis. *British Journal of Mathematical and Statistical Psychology, 44,* 315–331.

Dardanoni, V., & Forcina, A. (1998). A unified approach to likelihood inference or stochastic orderings in a nonparametric context. *Journal of the American Statistical Association, 93,* 1112–1123.

Dempster, A. P., Laird, N. M., & Rubin, D. B. (1977). Maximum likelihood estimation from incomplete data via the EM algorithm (with discussion). *Journal of the Royal Statistical Society, B Series, 39,* 1–38.

Gill, P. E., & Murray, W. (1974). *Numerical methods for constrained optimization.* London: Academic Press.

Goodman, L. A. (1974). Exploratory latent structure analysis using both identifiable and unidentifiable models. *Biometrika, 61,* 215–231.

Heinen, T. (1996). *Latent class and discrete latent trait models: Similarities and differences.* Thousand Oaks CA: Sage.

Hemker, B. T., Sijtsma, K., Molenaar, I. W., & Junker, B. W. (1997). Stochastic ordering using the latent trait and the sum score in polytomous IRT models. *Psychometrika, 62,* 331–347.

Hoijtink, H., & Molenaar, I. W. (1997). A multidimensional item response model: Constrained latent class analysis using the Gibbs sampler and posterior predictive checks. *Psychometrika, 62,* 171–190.

Lang, J. B., & Agresti, A. (1994). Simultaneously modeling the joint and marginal distributions of multivariate categorical responses. *Journal of the American Statistical Association, 89,* 625–632.

Langeheine, R., Pannekoek, J., & van de Pol, F. (1996). Bootstrapping goodness-of-fit measures in categorical data analysis. *Sociological Methods and Research, 24,* 492–516.

Lindsay, B., Clogg, C. C., & Grego, J. (1991). Semiparametric estimation in the Rasch model and related models, including a simple latent class model for item analysis. *Journal of the American Statistical Association, 86,* 96–107.

Masters, G. (1982). A Rasch model for partial credit scoring. *Psychometrika, 47,* 149–174.

Mellenbergh, G. J. (1995). Conceptual notes on models for discrete polytomous item responses. *Applied Psychological Measurement, 19,* 91–100.

Molenaar, I. W. (1997). Nonparametric models for polytomous responses. In W. J. van der Linden & R. K. Hambleton (Eds.), *Handbook of modern item response theory* (pp. 369–380). New York: Springer.

Muraki, E. (1992). A generalized partial credit model: Application of an EM algorithm. *Applied Psychological Measurement, 16,* 159–177.

Ritov, Y., & Gilula, Z. (1993). Analysis of contingency tables by correspondence models subject to order constraints. *Journal of the American Statistical Association, 88,* 1380–1387.

Robertson, T., Wright, F. T., & Dykstra, R. (1988). *Order restricted statistical inference.* New York: Wiley.

Samejima, F. (1969). Estimation of latent ability using a response pattern of graded scores. *Psychometrika Monograph Supplement No. 17.*

Sijtsma, K., & Hemker, B. T. (1998). Nonparametric polytomous IRT models for invariant item ordering, with results for parametric models. *Psychometrika, 63,* 183–200.

Tutz, G. (1990). Sequential item response models with an ordered response. *British Journal of Mathematical and Statistical Psychology, 43,* 39–55.

van der Ark, L. A. (2001). Relationships and properties of polytomous item response theory models. *Applied Psychological Measurement, 25,* 273–282.

Vermunt, J. K. (1993). *ℓEM: Log-linear and event history analysis with missing data using the EM algorithm* (WORC Paper No. 93.09.015/7). Tilburg, The Netherlands: Tilburg University.

Vermunt, J. K. (1997). *ℓEM: A general program for the analysis of categorical data* [Computer program]. Tilburg, The Netherlands: Tilburg University.

Vermunt, J. K. (1999). A general class of nonparametric models for ordinal categorical data. *Sociological Methodology, 29,* 187–224.

Software

The estimation procedures described are implemented in an experimental version of the ℓEM program (Vermunt, 1993, 1997).

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